Granular clustering: Self-consistent analysis for general coefficients of restitution

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We study the equilibrium behavior of one-dimensional granular clusters and one-particle granular gases for a variety of velocity-dependent coefficients of restitution r. We obtain equations describing the long-time behavior for the cluster's pressure, rms velocity, and granular interspacing. We show that for extremely long times, clusters with velocity-dependent coefficients of restitution are unstable and dissolve into homogeneous, quasielastic gases, but clusters with velocity-independent r are permanent. This is in accordance with hydrodynamic studies pointing to the transient nature of density instabilities for granular gases with velocity-dependent r.

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I. INTRODUCTION

Granular materials are present in many natural systems and play an important role in our daily lives [1,2] and in the economy since an estimated 1.3% of the U.S. electric power consumption goes into grinding particles and ores [3]. The interest in such systems ranges from the purely theoretical to daily practical applications, such as in the construction industry [2].

Typical granular systems (GS) are composed of large numbers of discrete macroscopic grains. Their shape is usually irregular, but in the present work we will consider them as smooth regular spheres of diameter d in vacuum, as a good approximation that still captures some of the essential physics of the problem. However, their dynamical and statistical properties may be affected by the presence of an interstitial medium, such as air or a liquid when their Bagnold number is small enough [2,4]. Granular materials behave in interesting ways exhibiting different features from ordinary solids, liquids, and gases, such as arches redistributing loads to the sides of solid arrangements of grains and inelasticity-induced nontrivial velocity distributions in rapidly flowing granular gases. Even some basic laws of thermodynamics, such as the zeroth law, may fail when extended to GS [5–7].

A very important aspect of the behavior of GS is their inherent tendency to cluster, i.e., a compaction due to an enhancement of the rate of collisions inside the system, accompanied by granular cooling down (kinetic energy reduction). Many authors have studied granular gas clustering, or compaction for denser systems. For GS, their behavior is initially homogeneous and starts with a given amount of kinetic energy [7–11], and they are subsequently left to cool down on their own. Their initial cooling behavior, the homogeneous cooling state (HCS), obeys Haff's law [11,12] for granular temperature (the typical internal average kinetic energy for the grains).

For longer times and rarefied granular systems, the intergranular collisions tend to correlate the motion (velocities and positions) of the grains, and techniques from kinetic theory will not be reliable anymore, at least in its simplest form [13–15]. Theoretical models have developed from this notion and obtained scaling forms for the transport coefficients [16,17] when the velocities of masses of grains become correlated inside a region of a certain characteristic length.

For clusters coalesced from smooth granular gases (no tangential restitution), there are no mechanisms for the exchange of angular momentum, and rolling does not occur (a totally irrelevant sliding motion between grains may still be present, but the rotational motion of the grains is not coupled to the translational motion if the grains are smooth). It is legitimate, from a theoretical point of view, to ask whether such structures are really permanently stable or if some other mechanism could lead the system again to gaseous homogeneity.

It has been known for quite some time that the hydrodynamic approximation predicts density instabilities for inelastic, smooth, hard-sphere granular systems at zero gravity [18] (velocity-independent coefficients of restitution), while for equivalent systems with velocity-dependent coefficient of restitution (r), such as the viscoelastic model [19], the instabilities are only transient [20]. Thus, the dependence of the coefficient of restitution on the velocity might be the cause of a possible cluster break-up. That is due to the fact that r tends to 1 as the impact relative velocity tends to zero. However, to simulate a cluster break-up can become computationally very costly, if no approximations are used. In most event-driven molecular-dynamics simulations, the coefficient of restitution has to be set to 1 as the relative velocity becomes smaller than an elastic threshold, so that collisions at relative velocities lower than the threshold will be elastic [11,20]. In order to be more realistic, one has to be able to reach extremely long times and small velocities. This is the goal of the present work. We will study a simple qualitative model (a one-dimensional cluster) and obtain the asymptotic solutions for its behavior at extremely long times, not yet accessible to computer simulations.

We study systems with coefficients of restitution given by the general form [21], at small relative velocity g,

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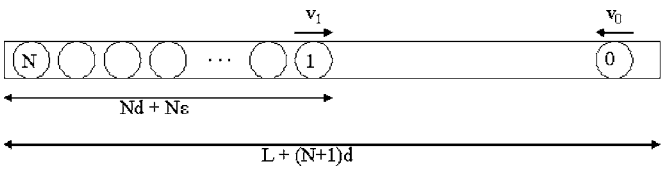

FIG. 1. The N particles of the cluster and the gas particle. The average distance between two consecutive particles of the cluster is ε (not shown), the diameter of each particle is d, and the total length is $L+(N+1)d$.

$$r = 1 - A\left(\frac{g}{g_0}\right)^m, \tag{1}$$

where $0 \leqslant A < 1$, and the constant g_0 sets the dissipation scale, and $m \geqslant 0$. The cases $m=0$ (velocity-independent r) and $m=1/5$ (viscoelastic) [19] are particular cases of Eq. (1).

We assume that the system starts as a granular gas inside a bounded one-dimensional container by walls at the extremities: one elastic and one inelastic. A cluster forms at the inelastic end and a gas still exists at the elastic side after some initial large time interval [15,22]. The granular gas portion will consist ultimately of a single grain. We will use this assumption in what follows, but this is not really essential; our results could easily be extended to multigrain gases, provided the gas kinetic energy goes asymptotically to zero.

We will obtain equations of motion for the dynamic and quasithermodynamic granular quantities: gas velocity, the cluster granular temperature (in fact its rms velocity), and internal pressure. We will show that the cluster is unstable at extremely long times, except for the velocity-independent case ($m=0$). Purely dynamic effects, due to energy dissipation, can be responsible for cluster formation but cannot hold it together indefinitely, except for the unrealistic velocity-independent coefficient of restitution case.

This paper is organized as follows. In Sec. II, we write the basic equations of collision dynamics. In Sec. III, we present our model and its corresponding equations. In Sec. IV, we study the stability conditions for the cluster. In Sec. V, the long-time behavior is extensively analyzed. In Secs. VI and VII, we discuss some of the consequences of the model and make our concluding remarks. The more technical aspects of this work are to be found in Appendixes A and B.

II. BASIC ASSUMPTIONS

Our approach is qualitative in nature and we do not expect it to reproduce the detailed behavior of a true granular cluster but only a few of its crudest properties.

Our model consists of an unforced one-dimensional system with $N+1$ identical smooth grains, with unit mass $M=1$, which can be modeled by inelastic hard spheres with velocity-dependent coefficient of restitution r, given by Eq. (1).

The grains are confined in a region of length $L+(N+1)d$ (d is the granular diameter) by an elastic wall on the right and an inelastic one on the left, as shown in Fig. 1. Relative to each other, the elastic wall presents some similarities to a “hot” wall while the inelastic one would be a “cold” one, as in Ref. [23], since the inelastic one is where the energy is taken out of the system. However, we are not injecting any amount of energy into the system and a steady state does not develop. The inelastic collisions at the inelastic wall are governed by the same Eq. (1), the wall being an infinitely heavy grain. The velocities of two colliding grains will be (V_1', V_2') after the collision, and (V_1, V_2) before it. They are related by

$$V_1' = \left(\frac{1-r}{2}\right)V_1 + \left(\frac{1+r}{2}\right)V_2, \tag{2}$$

$$V_2' = \left(\frac{1+r}{2}\right)V_1 + \left(\frac{1-r}{2}\right)V_2. \tag{3}$$

The system starts with a given initial amount of kinetic energy that will be dissipated due to the internal collisions. A partial clustering of grains will initially occur at the inelastic wall due to the pressure of the remaining gas [22]. The cluster phase is formed when the relative velocities are large compared with g_0 and the coefficient of restitution differs appreciably from 1. The gas will lose particles to the cluster until only one gas particle is left [15]. The difference between clustering and collapse is important, since in the latter an infinite number of collisions occurs among the particles in a finite amount of time. However, the cluster is only a very dense concentration of grains with small relative speeds. In one dimension, clustering can precede collapse (for r close to 1) [24].

After some initial interval of time, the velocities of all grains, that of the cluster and that of the gas, will be much smaller than the inelastic velocity scale g_0 in Eq. (1).

We label the gas particle as the zeroth particle while the cluster ones are labeled from 1 to N. We assume $g_0 \gg v_0 \gg v_{i=1,\ldots,N}$. Although the velocity of the gas particle is much larger than the velocities of the particles forming the cluster, the scaling factor g_0 will also be a lot larger than the velocity of the gas particle for long times. The following hierarchy for the small expansion parameters holds (for $m>0$),

$$\left|\frac{v_i}{v_0}\right| \gg \left|\frac{v_0}{g_0}\right|^m \gg \left(\frac{v_i}{v_0}\right)^2 \gg \left(\frac{v_i}{v_0}\right)\left|\frac{v_0}{g_0}\right|^m. \tag{4}$$

The logic of Eq. (4) is that at sufficiently long times $\frac{v_i}{g_0} \to 0$ and (since the gas will keep pumping energy into the cluster) the ratio v_i/v_0 will not vanish (that will be checked *a posteriori*, with respect to the long-time behavior of the cluster). We can assume (for $m>0$) that $|v_i/v_0| \gg |v_0/g_0|^m$. The second inequality comes from a choice that the time we choose to start our calculations is large but not too large, since for even larger times, certainly $|v_i/v_0|^2 \gg |v_0/g_0|^m$. The last inequality is a direct consequence of this choice of initial time. Higher-order terms shall be discarded.

We assume that our model has initial conditions satisfying Eq. (4).

III. GAS-CLUSTER EQUILIBRIUM

A. Gas pressure

In order to calculate the pressure exerted by the gas on the cluster, we need to take into account the momentum exchanged between the gas particle and the cluster after each collision between gas and cluster. A gas-cluster collision is completed when the gas particle leaves the cluster with an outgoing velocity (which is close in absolute value to its velocity before collision) much larger than the typical cluster's particle velocity. This is illustrated in Fig. 1. We notice that as the gas particle collides with the cluster, it is as if the "fast particle" (gas) would pass through the cluster's particles, collide with the inelastic wall, and go all the way back in the direction of the elastic wall. The process can be repeated until the "fast particle" crosses the whole cluster, back and forth. Thus, we obtain, to leading order, the momentum exchanged between the gas and cluster. This is calculated in Appendix A, Eq. (A6) (notice that V is the speed of the incoming gas particle, $V \geqslant 0$ always),

$$\dot{V} = -(2N+1)\frac{A}{4L}\left|\frac{V}{g_0}\right|^m V^2, \tag{5}$$

where the flipping of the "fast particle's" velocity at the inelastic wall is correctly taken into account in the final result above.

From Eq. (A7) ($m>0$), we obtain

$$p_{\text{gas}} = \frac{\Delta P_{\text{cluster}}}{\Delta t} = \frac{(1+m)}{2}\frac{\dot{V}}{V}\dot{\varepsilon}, \tag{6}$$

where p_{gas} is the pressure exerted by the gas on the cluster, and $\Delta P_{\text{cluster}}$ is the inelastic change of cluster momentum due to the collision with the gas particle.

For $m=0$, Eq. (A12) gives us

$$p_{\text{gas}} = \frac{1-(1-A)^N}{1+(1-A)^N}\dot{V}. \tag{7}$$

We will assume in the following that the product NA is small. Our results, being valid for small NA in the $m=0$ case, will show that collapse happens in a finite amount of time. This will certainly be valid for the case of large NA.

B. Cluster variables

Other important variables exist that we need to take into account. The variance of the cluster's particle velocities σ^2 is one of them. It is defined as

$$\sigma^2 = \frac{\sum_{i=1}^{N}(v_i - v_{\text{c.m.}})^2}{N}, \tag{8}$$

where the cluster's center-of-mass velocity is given by

$$v_{\text{c.m.}} = \frac{\sum_{i=1}^{N} v_i}{N}. \tag{9}$$

Another such variable is the mean granular spacing, ε. It is defined by (with $x_{N+1}=0$)

$$\varepsilon = \frac{\sum_{i=1}^{N+1}|x_i - x_{i-1} - d|}{N}. \tag{10}$$

The cluster's full size is thus $N(d+\varepsilon)$. A somewhat crude, but reasonable, approximation for the cluster's center-of-mass velocity can be obtained by assuming that the cluster expands or contracts rather uniformly (at least for small ε), and thus

$$v_{\text{c.m.}} \approx \sum_{i}^{N} Ni\dot{\varepsilon} = \frac{N(N+1)}{2N}\dot{\varepsilon}. \tag{11}$$

In the limit of large N, the expression above reduces to

$$v_{\text{c.m.}} \approx \frac{N}{2}\dot{\varepsilon}.$$

The center-of-mass acceleration then reads

$$a_{\text{c.m.}} = \frac{N(N+1)}{2N}\ddot{\varepsilon}. \tag{12}$$

C. Wall pressure and mean spacing

The cluster feels two external sources of pressure: the pressure from the gas and the pressure due to interactions with the inelastic wall. That pressure can be calculated using a crude approximation by assuming that the momenta exchanged between the particle labeled N at every collision is of the order of 2σ and the rate of collision is σ/ε [25]. The mean-field wall pressure is then

$$p_{\text{wall}} = \frac{2\sigma^2}{\varepsilon} > 0. \tag{13}$$

The equation of time evolution for the mean spacing ε is obtained by writing Newton's law for the cluster (p_{wall} is a positive force while p_{gas} is a negative one),

$$a_{\text{c.m.}} = p_{\text{wall}} + p_{\text{gas}}.$$

Using Eqs. (6), (12), and (13), we obtain the mean-field equation for $m>0$,

$$\frac{N}{2}\ddot{\varepsilon} = \frac{2\sigma^2}{\varepsilon} + \frac{(1+m)}{2}\frac{\dot{V}}{V}\dot{\varepsilon}. \tag{14}$$

The equivalent equation for the case $m=0$ is obtained from p_{gas} given by Eq. (A12),

$$\frac{N}{2}\ddot{\varepsilon} = \frac{2\sigma^2}{\varepsilon} + \frac{1-(1-A)^N}{1+(1-A)^N}\dot{V}. \tag{15}$$

D. Energy dissipation inside the cluster

A calculation similar to that for the gas velocity reduction is done in Appendix B for the energy of the cluster, due to the effect of gas-cluster collision. At the order of approximation we have set, $|V/g_0|^m$, the corrections will be of the order $\sigma^2|V/g_0|^{2m}$ (much smaller than σ^2/ε). The comparison of the cluster's kinetic energy before and after a collision with the gas particle then reads

$$\sum_{i}^{N}(v_i'''')^2 = \sum_{i}^{N}(v_i)^2. \tag{16}$$

The cluster's kinetic energy is not affected by the gas-cluster collision, at the order of approximation used. This tells us that only internal collisions will be important for cooling down the cluster.

In a mean-field approximation, the energy dissipated per particle corresponds to the product of the energy lost in each collision and the rate of collision per particle. For an N-particle one-dimensional gas confined in a free volume, $l_{1\mathrm{D}}$, with the typical velocity variance σ (v is the typical velocity, of the order of the square root of the variance σ), the energy loss per collision corresponds roughly, in dimensionless terms, to

$$\Delta v \propto -|v|^{1+m}.$$

The rate of collision is proportional to $v/l_{1\mathrm{D}}$ [25]. So, in order to calculate the rate of dissipation of energy inside the cluster, with typical interspacing becoming very small as $t \to \infty$, but not yet zero, we need to estimate the internal collision rate and multiply it by the loss of energy for each collision. The rate of collisions for a cluster only differs from that of a gas because its interspacing ε is very small compared to L. Taking it into account, the cluster's collision rate is now obtained as

$$q = N\frac{\sigma}{2\varepsilon}.$$

The variation of the cluster's typical internal velocity is due to the collision between grains. We assume that colliding grains will have relative velocities of the order σ and the quasielastic collisions (in the limit $g \to 0$) will switch those velocities with a small loss,

$$\Delta\sigma = -A\left(\frac{\sigma}{g_0}\right)^m \sigma.$$

The change in σ per unit time is the product $q\Delta\sigma$. Thus we obtain the heuristic equation

$$\dot{\sigma} = -\frac{NA\sigma^2}{2\varepsilon}\left(\frac{\sigma}{g_0}\right)^m. \tag{17}$$

The equivalent form for $m=0$ is derived from Eq. (B7) from Appendix B,

$$\dot{\sigma} \sim -\frac{\sigma^2}{\varepsilon}. \tag{18}$$

IV. DIMENSIONLESS ANALYSIS

We want to study, qualitatively, the conditions under which the cluster will be stable, and we shall use the approximate equations of motion obtained above. However, we will not be interested in the fine details of the equations themselves, only in their asymptotic behavior in time. Then, we will rewrite Eqs. (5), (14), (15), and (17) in a completely dimensionless form (notice that $V, \sigma, \varepsilon > 0$ for all t). First, the equations for V and σ have the same form for both $m=0$ and $m>0$. They read

$$\dot{V} = -V^{2+m}, \tag{19}$$

$$\dot{\sigma} = -\frac{\sigma^{2+m}}{\varepsilon}. \tag{20}$$

The equation for $\ddot{\varepsilon}$ for $m=0$ reads

$$\ddot{\varepsilon} = \frac{\sigma^2}{\varepsilon} + \dot{V} \tag{21}$$

and that for $m>0$ reads

$$\ddot{\varepsilon} = \frac{\sigma^2}{\varepsilon} + \frac{\dot{V}}{V}\dot{\varepsilon}. \tag{22}$$

In order to recover the dimensional units, remember that σ and V are given in terms of g_0, and ε is measured in terms of L/N. A few dimensional constants have to be used in Eqs. (19)–(22) in order to make both sides dimensionally coherent.

Equation (19) can be exactly solved, obtaining

$$V = \frac{V_0}{[1+(1+m)V_0^{1+m}t]^{1/(1+m)}}. \tag{23}$$

This is the extension of Haff's law [12,26] to the cases described by Eq. (1). We observe that as

$$t \to \infty \Rightarrow V \sim t^{-1/(1+m)} \Rightarrow T_g \sim t^{-2/(1+m)}.$$

For the case of velocity-independent coefficient of restitution, $m=0$, $T_g \sim t^{-2}$. For the viscoelastic coefficient of restitution case [19], $m=1/5$, $T_g \sim t^{-5/3}$ as expected [26]. Equations (20)–(22) will constitute the system to be solved in the following.

V. LONG-TIME BEHAVIOR

We must keep in mind that there is an implicit velocity scale g_0 that divides the velocity variables whenever a power of m comes into play (a consequence of the form of the coefficient of restitution). For the constant coefficient of restitution case, we obtain a useful equation by combining Eq. (20) with Eq. (21),

$$\dot{\varepsilon} - V + \frac{\sigma^{1-m}}{1-m} = c_0, \tag{24}$$

where the constant c_0 is related to the initial conditions for V, σ, and ε by

$$c_0 = \dot{\varepsilon}_0 - V_0 + \frac{\sigma_0^{1-m}}{1-m}. \tag{25}$$

The equations describing the granular cluster are valid in the limit when $V \gg \sigma, \dot{\varepsilon}$. In the following, we analyze the cluster behavior for the whole range of values of m based on Eqs. (19)–(24). The figures are obtained by solving numerically Eqs. (19)–(24).

A. $m=0$

In this case, the coefficient of restitution does not depend on the impact relative velocity. We have

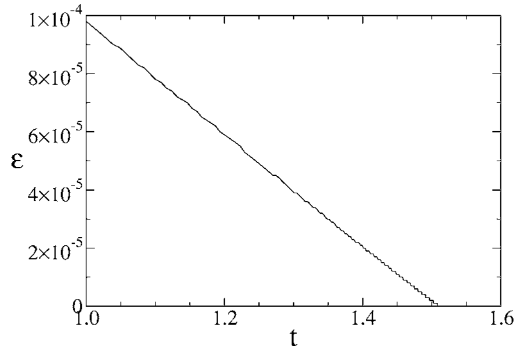

FIG. 2. Coefficient of restitution independent of the velocities. The only stable case at all times. The mean spacing ε is measured in units of L/N.

$$c_0 = \dot{\varepsilon}_0 - V_0 + \sigma_0 \approx -V_0, \tag{26}$$

$$\dot{\varepsilon} = V - \sigma - V_0. \tag{27}$$

As $t \to \infty$, we observe that V and σ tend to zero [27] and thus $\dot{\varepsilon} \approx -V_0 < 0$.

In practice, it is impossible to observe the asymptotic limit above since the collapse happens in a finite amount of time (see Fig. 2). The cluster is thus stable and will not dissolve itself.

B. $m > 0$

When the coefficient of restitution depends on the initial relative velocity, i.e., $m > 0$, we can see that the physical behavior of the system changes qualitatively, as shown in Appendix A.

A scaling argument can be used and compared with the result of simulations in order to obtain the very long-time behavior of the variables V, σ, and ε. The asymptotic solutions for ε and σ can be written as powers of time and log-time,

$$V \sim t^{\beta_1}, \quad \sigma \sim t^{\beta_2}(\ln t)^{\alpha_2}, \quad \varepsilon \sim t^{\beta_3}(\ln t)^{\alpha_3}, \tag{28}$$

where we have already determined β_1,

$$\beta_1 = -\frac{1}{1+m}.$$

The solutions for α_2, β_2, α_3, and β_3 are

$$\alpha_2 = -\frac{1}{m}, \tag{29}$$

$$\beta_2 = 0, \tag{30}$$

$$\alpha_3 = -\frac{1}{m}, \tag{31}$$

$$\beta_3 = 1. \tag{32}$$

Thus, the long-time behavior of σ and ε is given by

$$\sigma \sim (\ln t)^{-1/m} \quad \text{and} \quad \varepsilon \sim t(\ln t)^{-1/m}. \tag{33}$$

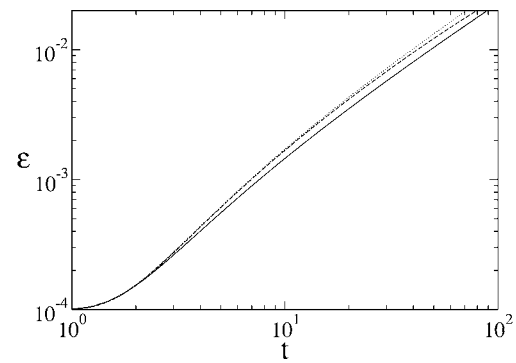

FIG. 3. Time evolution for a few examples of velocity-dependent coefficients of restitution: full line, $m=0.2$ (viscoelastic model); dashed line, $m=1$; dotted line, $m=2$.

These are self-consistent, logarithmically corrected solutions for σ and ε at long times. However, we need to look into the long-time behavior of σ in more detail.

For $m > 0$, there is no granular collapse. After some transient time, the cluster will grow almost linearly (as can be seen in Fig. 3, consistent with the main behavior of $\varepsilon \sim t$) and will eventually occupy the whole container, in fact becoming once again a granular gas with interspacing $\varepsilon \sim L/N$. It takes an enormous amount of time for this to happen. This is illustrated in Fig. 3, where we compare the cases $m=0.2$, $m=1$, and $m=2$.

At long times, our model becomes quasielastic, to a very good approximation, when $m > 0$. For the velocity-dependent case, the internal dissipation for the cluster becomes negligible ($\beta_2=0$) but its internal energy is not conserved. The apparent contradiction between $\beta_2=0$ and $\sigma \to 0$ does not hold since σ decays as a power of $\ln t$. Even more significantly, in our model $\varepsilon \leqslant L/N$ and the growth of ε has to be cut off correspondingly. Since our mean-field equations do not impose a boundary to ε, Eq. (20) will give us $\dot{\sigma} \to 0$ as $t \to \infty$, consistent with $\beta_2=0$. In reality, after reaching the cutoff size, normal gas dissipation takes over and the former cluster will follow Haff's law for energy dissipation again.

Another important consistency argument can be extracted from Eq. (33). If we take the limit $m \to 0$ before the limit $t \to \infty$ is taken, we observe that $\varepsilon = \sigma = 0$ results. This is in complete agreement (remember that the initial time is taken to be long for the $m > 0$ case) with our result that a granular collapse happens in a finite interval of time when $m=0$.

It is interesting to notice at this point that in Ref. [20] the authors simulated a velocity-dependent granular system with an elastic threshold ($r=1$ below a certain threshold relative velocity) and supposed their results to be extensible to the viscoelastic regime. This is in accordance with our results. However, the form of the coefficient of restitution in Ref. [20] mimics much more closely the $m > 1$ case than the $m=0.2$ case. As we observe in our calculations, ε will neither tend to zero nor remain stable in both cases, which is consistent with the results in Ref. [20].

C. Oscillations

An interesting feature we have observed is very low frequency size oscillations at very long times. Equation (20)

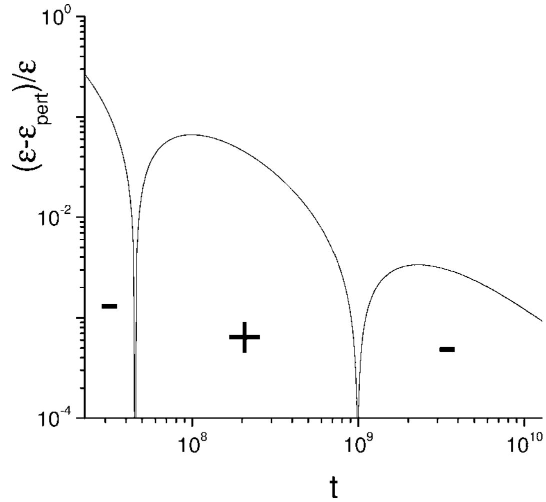

FIG. 4. Plot of the unperturbed value of ε minus the perturbed one, as a function of time, normalized by ε itself. It is consistent with Eq. (34), a damped-harmonic-oscillator-like equation. The sign corresponds to whether the oscillation has a positive or a negative value, since the plot corresponds to the logarithm of the absolute value of the difference between the two freely evolving solutions.

predicts oscillations with decreasing frequency. We can study the case of a small perturbation on ε such as

$$\varepsilon = t(\ln t)^{-1/m}[1+\phi].$$

We obtain, after some straightforward algebra, the asymptotic equation for the relative perturbation ϕ,

$$\ddot{\phi} + \frac{\ln t}{t}\dot{\phi} + \frac{2\ln t}{t^2}\phi = 0. \tag{34}$$

It is similar to a low frequency damped harmonic oscillator, with a frequency that goes to zero as $t^{-1}(\ln t)^{1/2}$.

The effect of small perturbations in the asymptotic value of ε is rather hard to observe directly. However, we observed it by initially running our simulations in order to obtain aged values of ε, σ, and its derivatives. We then perturb ε as $(1+\Delta)\varepsilon$, with $\Delta=1.0\times10^{-4}$.

We run two subsequent calculations, with an aged and unperturbed solution as the initial condition, and another for the perturbed one. Their difference should also obey Eq. (34). The result is plotted in Fig. 4, on a logarithmic scale (we plot the absolute value of ϕ; the signs correspond to whether ϕ is positive or negative).

It can be seen that the period is indeed increasing [it is of the order of the total time, consistent with a "frequency" of order $t^{-1}(\ln t)^{1/2}$].

VI. CONSEQUENCES OF THE MODEL

The most immediate consequence of the present model (for $m\neq0$) is the evidence it provides of the transient nature for some of the granular singularities in a freely cooling granular gas with a velocity-dependent coefficient of restitution. This indicates that a hydrodynamic treatment might be adequate for such systems, at least after a transient time. Also, we deduce from our results that purely dynamical effects cannot give rise to permanent clusters if $m\neq0$ at the zero-energy feeding regime.

Another consequence is the eventual evaporation of clusters for smooth granular systems. The inviscid Burgers' equation has been proposed as a mechanism of formation for a granular cluster with velocity-independent coefficient of restitution [11,28]. For systems with $m\neq0$, one may ask whether that equation is still adequate, and what kind of regime might replace it, in the evaporative period (at extremely long times). Work is currently underway along this direction.

The noncollapse when $m\neq0$ gives us hope that it might be possible to treat two- or three-dimensional clusters as very dense, but nonsingular, granular phases (for smooth systems) describable by internal, nondiverging, variables (maybe even similar ones to the σ and ε used in this manuscript). That could make it easier to incorporate the treatment of clusters into the hydrodynamic methods available today.

VII. CONCLUSIONS

We study the long-term stability of unforced granular systems, in which clusters form, with the help of a qualitative, microscopic model that makes it possible to look at clusters at extremely long times, not available to computer simulations.

We assume a general form for the coefficient of restitution that includes the well known velocity-independent and viscoelastic models as special cases.

We are interested in this problem for two main reasons. First, despite its apparent simplicity, a granular cluster's behavior, at extremely long times, depends on the amount of inelasticity (which can be defined as $q=\frac{1-r}{2}$). According to our model, if the coefficient of restitution becomes 1 as the relative velocity of impact tends to zero, as with most realistic systems, then clusters of rigid, smooth spheres will be unstable (at least at zero gravity). This suggests a rich dynamical behavior for our granular gas that comprises an initial homogeneous phase in which Haff's law [12] predicts the evolution of the average granular temperature. The system goes into phase separation after a transient time and the global kinetic energy varies with a different power of time [11]. After a very long waiting time, the external granular gas pressure no longer keeps the cluster particles together and the cluster finally dissolves into an extremely slow moving homogeneous granular gas. Haff's law will once again apply to this gas (since $m\neq0$). This is not in contradiction with the results in Ref. [28] since the results therein apply to systems with velocity-independent coefficients of restitution ($m=0$).

Secondly, for velocity-dependent coefficients of restitution, the clusters are not truly collapsed, but behave instead as very dense, fluid phases (for zero surface friction and zero gravity). In fact, we could think of the gas-cluster phase coexistence boundary as a smooth separation between the granular gas and cluster phases, without a singular boundary, except for the case of constant coefficients of restitution ($m=0$). An appropriate continuous hydrodynamic treatment for it might be possible.

Questions arise concerning the long-time dissolution of granular clusters: will they obey the same equations as the ones that are found to apply for the collapsing phase? Since the irreversibility of the "microscopic," e.g., granular, dynamics prevents time reversal from applying, the dissolution equations might be quite different from the collapse ones. This is yet to be understood.

APPENDIX A: GAS-CLUSTER MOMENTUM EXCHANGE

1. $m>0$

From Eqs. (2) and (3) we obtain the postcollision velocities for two particles of the same mass (the collision time is taken to be zero),

$$v_1' = \left(1 - \frac{A}{2}\left|\frac{v_1 - v_0}{g_0}\right|^m\right)v_0 + \frac{A}{2}\left|\frac{v_1 - v_0}{g_0}\right|^m v_1, \quad \text{(A1)}$$

$$v_0'' = \frac{A}{2}\left|\frac{v_1 - v_0}{g_0}\right|^m v_0 + \left(1 - \frac{A}{2}\left|\frac{v_1 - v_0}{g_0}\right|^m\right)v_1. \quad \text{(A2)}$$

The final velocity of the "fast particle" after crossing the cluster can be calculated by the equations of basic collision dynamics. Assuming $g_0 \gg |v_0| = V \gg |v_1|$, we can expand the last term of Eq. (1) as follows:

$$\left|\frac{v_1 - v_0}{g_0}\right|^m = \left|\frac{v_0}{g_0}\right|^m \left|1 - \frac{v_1}{v_0}\right|^m \cong \left|\frac{V}{g_0}\right|^m \left(1 + m\frac{v_1}{V}\right). \quad \text{(A3)}$$

The velocities of particle 1 and the gas particle, after the first collision, can be rewritten with the help of Eqs. (A1) and (A2) as

$$v_1' \approx -V + \frac{A}{2}\left|\frac{V}{g_0}\right|^m V + (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m v_1,$$

now being the fast particle, and the gas one is now slow (primes stand for fast-slow collisions),

$$v_0'' \approx v_1 - \frac{A}{2}\left|\frac{V}{g_0}\right|^m V - (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m v_1.$$

After ℓ collisions, the fast particle velocity will be the ℓth one,

$$v_\ell' \approx -V + \ell\frac{A}{2}\left|\frac{V}{g_0}\right|^m V + (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m \sum_{i=1}^{\ell-1} v_i,$$

and the $(\ell-1)$th particle (which suffered two collisions) has the velocity

$$v_{\ell-1}'' \approx v_\ell - \frac{A}{2}\left|\frac{V}{g_0}\right|^m V - (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m v_\ell.$$

After colliding N times, the fast particle will reach the inelastic wall. Its velocity, prior to that collision, will then read

$$v_N' = -V + N\frac{A}{2}\left|\frac{V}{g_0}\right|^m V + (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m p_{\mathrm{cl}},$$

where $p_{\mathrm{cl}} = \Sigma_{i=1}^N v_i$, the cluster's total momentum before colliding with the gas.

Hence the total momentum given by the gas to the cluster (first part) is

$$\Delta p_{\mathrm{cluster}_1} = -N\frac{A}{2}\left|\frac{V}{g_0}\right|^m V - (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m p_{\mathrm{cl}}.$$

After the collision with the inelastic wall, it reads

$$V' = v_N'' = V - (N+1)\frac{A}{2}\left|\frac{V}{g_0}\right|^m V - (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m p_{\mathrm{cl}}.$$

The procedure for the calculations of how the fast particle traverses the cluster is similar to the one above and the final result is (up to the same approximation order)

$$v_0''' = V' - N\frac{A}{2}\left|\frac{V}{g_0}\right|^m V' + (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m p_{\mathrm{cl}}'',$$

where $p_{\mathrm{cl}}'' = \Sigma_{i=1}^N v_i''$.

The momentum received by the cluster due to these collisions is then

$$\Delta p_{\mathrm{cluster}_2} = N\frac{A}{2}\left|\frac{V}{g_0}\right|^m V' - (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m p_{\mathrm{cl}}'',$$
$$= N\frac{A}{2}\left|\frac{V}{g_0}\right|^m V - (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m p_{\mathrm{cl}}''.$$

Before the collision with the elastic wall, the fast particle velocity is given by

$$v_0''' = V' - N\frac{A}{2}\left|\frac{V}{g_0}\right|^m V' + (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m p_{\mathrm{cl}}'',$$
$$= V - (2N+1)\frac{A}{2}\left|\frac{V}{g_0}\right|^m V + (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m (p_{\mathrm{cl}}'' - p_{\mathrm{cl}}).$$

After that last collision, the gas particle has the velocity

$$v_0'''' = -V + (2N+1)\frac{A}{2}\left|\frac{V}{g_0}\right|^m V - (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m (p_{\mathrm{cl}}'' - p_{\mathrm{cl}}).$$

Hence, the absolute value of the gas particle velocity varies (for a single gas-cluster collision cycle) as

$$\Delta V = -(2N+1)\frac{A}{2}\left|\frac{V}{g_0}\right|^m V + (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m (p_{\mathrm{cl}}'' - p_{\mathrm{cl}}),$$
$$= -(2N+1)\frac{A}{2}\left|\frac{V}{g_0}\right|^m V, \quad \text{(A4)}$$

where we discarded terms of the order $O(|V/g_0|^{2m})$.

Thus, the total momentum absorbed by the cluster from the gas-cluster collision is given by

$$\Delta p_{\rm cl\ tot} = \Delta p_{{\rm cluster}_1} + \Delta p_{{\rm cluster}_2} = -(1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m (p''_{\rm cl} + p_{\rm cl})$$
$$= -(1+m)A\left|\frac{V}{g_0}\right|^m p_{\rm cl}. \qquad \text{(A5)}$$

The reader should notice that we assume the initial time to be large enough so that quantities such as $NA|V/g_0|^m$ are small and the total dissipation per gas-cluster collision can be a small fraction of the gas kinetic energy.

Equations (A4) and (A5) are the fundamental result of this appendix. We can transform them into rate equations by determining the rate of gas-cluster collisions. The time interval between successive collisions is given by $\Delta t = 2L/V$.

We obtain the equation governing the behavior of the absolute value of the gas velocity,

$$\dot{V} \equiv \frac{\Delta V}{\Delta t} = -(2N+1)\frac{A}{4L}\left|\frac{V}{g_0}\right|^m V^2. \qquad \text{(A6)}$$

The equation for the gas pressure, the rate of transfer of momentum, is also obtained,

$$p_{\rm gas} \equiv \frac{\Delta p_{\rm cl\ tot}}{\Delta t} = -(1+m)\frac{A}{2L}\left|\frac{V}{g_0}\right|^m V p_{\rm cl}.$$

Notice that for a large cluster, $N \gg 1$, we can write

$$p_{\rm gas} = \frac{(1+m)}{2}\frac{\dot{V}}{V}\dot{\varepsilon}. \qquad \text{(A7)}$$

2. $m=0$

The case of a constant coefficient of restitution deserves a separate treatment. In this case, we assume that $A \ll 1$ and $r = 1 - A \approx 1$.

After a collision with a slow particle, the fast particle acquires a velocity $v' = (1-A)v$. At the end of a sequence of N such collisions, the velocity of the fast particle (before colliding with the inelastic wall) will be

$$v_N = (1-A)^N v_0.$$

The momentum exchanged with the cluster is then

$$\Delta p_{c1} = [(1-A)^N - 1]v_0.$$

After colliding with the inelastic wall, the gas particle has a velocity $v_N = -(1-A)^N v_0$. After colliding another N times with cluster particles, the gas particle velocity will be

$$v''_N = -(1-A)^{2N} v_0.$$

The momentum exchanged with the cluster this time is then

$$\Delta p_{c2} = -(1-A)^N[(1-A)^N - 1]v_0.$$

The total change in velocity for the gas particle, after collision with the elastic wall, is given by

$$\Delta V = -[1 - (1-A)^{2N}]V. \qquad \text{(A8)}$$

The rate of change of V is given (see the calculation for $m > 0$ above),

$$\dot{V} = -\left(\frac{1-(1-A)^{2N}}{2L}\right)V^2. \qquad \text{(A9)}$$

The total momentum gained by the cluster after the collision is then

$$\Delta p_c = \Delta p_{c1} + \Delta p_{c2} = -[(1-A)^N - 1]^2 V. \qquad \text{(A10)}$$

The gas pressure in this case will be given by (similar to the case for $m > 0$)

$$p_{\rm gas} = \frac{\Delta p_c}{\Delta t} = -\frac{[(1-A)^N - 1]^2}{2L}V^2. \qquad \text{(A11)}$$

We can see that the gas pressure is related to the change in gas velocity through

$$p_{\rm gas} = \frac{1-(1-A)^N}{1+(1-A)^N}\dot{V}. \qquad \text{(A12)}$$

There is a clear change in the gas pressure regime for $m > 0$ compared with the more commonly used case of $m=0$. This makes the pressure applied by the gas weaker since, for $m > 0$, the factor $\dot{V}$ [see Eq. (A12)] is multiplied by a factor $\dot{\varepsilon}/V$ [see Eq. (A7)].

APPENDIX B: CLUSTER ENERGY DISSIPATION IN THE GAS-CLUSTER COLLISION

1. $m > 0$

In order to show that the cluster's kinetic energy is not affected by the gas-cluster collision on the order of approximation we have chosen, let us consider the sum of the velocities after the first passage of the gas particle all the way to the inelastic wall,

$$\sum_{i=0}^{N-1} v''_i = \left[1 - (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m\right]\sum_{i=1}^{N} v_i - \frac{A}{2}\left|\frac{V}{g_0}\right|^m NV. \qquad \text{(B1)}$$

Our calculations are carried out to order $N|V/g_0|^m$ in the development of the coefficient of restitution, in the spirit of Eq. (4). We suppressed terms coming from orders smaller than $N|V/g_0|^m$.

Let us also consider the sum of the velocities after the passage back of the gas particle,

$$\sum_{i=1}^{N} v''''_i = \left[1 - (1+m)\frac{A}{2}\left|\frac{V}{g_0}\right|^m\right]\sum_{i=0}^{N-1} v''_i - \frac{A}{2}\left|\frac{V}{g_0}\right|^m NV. \qquad \text{(B2)}$$

These equations can be added up giving

$$\sum_{i=1}^{N} v''''_i - \sum_{i=1}^{N} v_i = -(1+m)A\left|\frac{V}{g_0}\right|^m \sum_{i=1}^{N} v_i, \qquad \text{(B3)}$$

yielding the pressure exerted on the cluster by the gas.

We can proceed along similar lines for the sum of square velocities and obtain

$$\sum_{i=0}^{N-1}(v_i'')^2 = \sum_{i=1}^{N} v_i^2 - AV\left|\frac{V}{g_0}\right|^m \sum_{i=1}^{N} v_i \tag{B4}$$

and

$$\sum_{i=1}^{N}(v_i'''')^2 = \sum_{i=0}^{N-1}(v_i'')^2 + AV\left|\frac{V}{g_0}\right|^m \sum_{i=0}^{N-1} v_i''. \tag{B5}$$

Equations (B1), (B4), and (B5) show that the kinetic energy of the cluster remains the same,

$$\sum_{i=1}^{N}(v_i'''')^2 = \sum_{i=1}^{N} v_i^2 + O(\sigma^2|V/g_0|^{2m}). \tag{B6}$$

2. $m=0$

As shown in Appendix A 2, the gas grain pumps momentum into the cluster. We will assume that $NA \ll 1$. This is not too restrictive to our argument since we will show that a long-time granular collapse happens for the quasielastic velocity-independent coefficient of restitution case. Thus, it will certainly happen for the case when NA is large too.

We noticed that as the fast grain collides with the cluster, it gives energy to it by changing the particles' velocities by an amount proportional to $NA\dot{V}$. After squaring all cluster particles' velocities (relative to the center of mass of the cluster), adding them all up, and subtracting the initial value of it, we obtain a rate of energy, pumped into the cluster, proportional to the product of $\dot{V}$ and $\dot{\varepsilon}$.

The rate of change of σ has two main contributions: a negative one from internal collisions and a positive one from gas-cluster collision. The second one is negligible and we do not take it into account further. The reason for this is as follows. Since $|\dot{V}| \gg |\dot{\varepsilon}\dot{V}|$, if we assume $|\dot{\varepsilon}\dot{V}| > \sigma^2/\varepsilon$, then σ will decay more slowly when the energy pumping term is present but the wall pressure term in Eq. (21) will still be much smaller than the gas one. However, we can check *a posteriori* that even when the energy pumping term is not present, the mean interspacing falls at a linear rate in a finite time (collapse, see Fig. 2). It yields

$$\sigma^2/\varepsilon \sim |\dot{V}| \gg |\dot{V}\dot{\varepsilon}|.$$

Thus, we only keep the internal collisions dissipation term in the equation for $\dot{\sigma}$,

$$\dot{\sigma} \approx -\frac{\sigma^2}{\varepsilon}. \tag{B7}$$

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[27] From Eqs. (19) and (20), we can write $\frac{dV^{-1-m}}{dt}=\frac{1+m}{L}$ and $\frac{d\sigma^{-1-m}}{dt}=\frac{1+m}{\varepsilon}=\frac{1+m}{L}\frac{L}{\varepsilon}\gg\frac{1+m}{L}$. Hence, σ^{-1-m} grows much faster than V^{-1-m}, so σ goes to zero faster than V.

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